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## TWO GENERALIZATIONS OF THE STIELTJES INTEGRAL.

By P. J. DANIELL.

1. **Introduction.** In a paper on a General Form of Integral\* the author gives an example of an integral with respect to a function which is not of limited variation, namely,

$$\int_0^1 f(x) d \log x,$$

which can be defined when  $f(x)/x$  is continuous ( $0 \leq x \leq 1$ ). The first part of the present paper is an extension to a general class of integrals of this type. In other words, it considers integrals with respect to a general function  $\alpha(x)$  which can be defined when appropriate restrictions are laid on the integrand  $f(x)$ .

The Stieltjes integral differs from the more usual integral in that it is invariant under a transformation of the independent variable which leaves relative position unchanged if the mass-distribution is transformed in the corresponding manner. It is less dependent on metrical geometry. This suggests an extension of the concept to an integral which is an operation on sets directly without any interpolation of measure. This concept may be useful in the theory of sets of points, but apart from that it opens an interesting field.

2. **Integration with respect to any function.** The notion we are about to develop can be extended to several dimensions but we prefer to give the development only for a single variable. Let  $\alpha(x)$  be some function defined for all real values of  $x$ . If it is not, we may extend its definition by assigning to it the value 0 wherever it is not defined.

Relative to  $\alpha(x)$  a point  $x$  is said to be proper on the right if an interval  $xx'$  can be found of which  $x$  is the left-hand endpoint and in which  $\alpha(x)$  is of limited variation. It is improper on the right if such an interval cannot be found. Similarly it is proper (improper) on the left if  $\alpha(x)$  is of limited variation in some (no) interval  $x''x$  to the left of  $x$ . Every point is either proper or improper on the right and also either proper or improper on the left. If a point is proper on both sides, it is said to be "proper" (without qualification), while if it is improper on either side, it is said to be "improper." If further it is improper on both sides, we may say that it is "completely improper."

**THEOREM.** *If  $p_1, p_2, \dots$  is an increasing sequence of improper points and if the limit of the sequence is  $p$ ,  $p$  is at least improper on the left.*

\* P. J. Daniell, these Annals, vol. 19 (1918), p. 279.

For if not, an interval  $p'p$  could be found to the left of  $p$  such that in it  $\alpha$  is of limited variation. But such an interval would enclose a point  $p_n$  of the sequence so that  $\alpha$  is of limited variation in both  $p'p_n$  and  $p_np$ . This contradicts the hypothesis that each  $p_n$  is improper. Similarly if a point  $p$  is approached by a sequence of improper points from the right,  $p$  is improper on the right.

It follows that the set of improper points is *closed* in the sense of the theory of sets (e.g., the set of all non-negative numbers is closed but not compact). Again any proper point lies strictly within an interval of proper points. Hence the set of proper points consists of a countable (that is, zero, finite, or denumerably infinite) set of non-compact intervals,  $\delta_n$ , the complement of the closed set of improper points,  $K$ . A non-compact interval is usually an open interval but it is an open question whether the interval consisting of all real numbers should be called an open interval or not. As a set of points it is closed since it includes its derived set.

Every point which is improper on the left only is the left-hand endpoint of some interval  $\delta_n$ , for otherwise it would be the limit from the right of a sequence of improper points; and similarly for points improper to the right only. Hence the set of points which are improper on one side only is countable. A point of  $K$  which is not an endpoint of an interval  $\delta$  is completely improper. But an endpoint may also be completely improper. For example, let  $\alpha(x)$  be defined as

$$\begin{aligned}\alpha(x) &= \sin 1/x & x \neq 0, \\ &= 0 & x = 0.\end{aligned}$$

Then the proper intervals are  $(-\infty < x < 0)$ ,  $(0 < x < +\infty)$ .  $x = 0$  is the endpoint of two intervals  $\delta$ , but it is completely improper. On the other hand if

$$\begin{aligned}\alpha(x) &= \sin 1/x & x > 0, \\ &= 0 & x \leq 0,\end{aligned}$$

the proper intervals are as before but  $x = 0$  is improper to the right only.

Let  $\Delta$  be a closed and compact interval contained strictly within an interval  $\delta_n$ . Then  $\alpha(x)$  must be of limited variation on  $\Delta$ . For every point of  $\Delta$  is strictly within an interval in which  $\alpha$  is of limited variation. By the Heine-Borel theorem a finite number of these intervals can be found covering  $\Delta$  completely between them so that  $\alpha(x)$  is of limited variation over their sum, which includes  $\Delta$ .

It follows that there can only be a countable number of points in any  $\Delta$  (and therefore in  $\Sigma\delta$ ) at which  $\alpha(x)$  is discontinuous. Let  $\Delta = (x_1, x_2)$  be an interval enclosed strictly within a  $\delta_n$  and such that  $\alpha(x)$  is continuous

at  $x_1, x_2$ . We define the "mass" of  $\Delta$  as

$$m(\Delta) = \alpha(x_2) - \alpha(x_1).$$

Let  $\varphi_n(x)$  be a function equal to 1 on such an interval  $\Delta_n$  and 0 elsewhere. We define

$$\int \varphi_n(x) d\alpha(x) = m(\Delta_n).$$

If  $f(x)$  is a linear combination of a finite number of such functions  $\varphi_n(x)$ ,

$$f(x) = c_1\varphi_1(x) + \cdots + c_n\varphi_n(x),$$

we define

$$\int f(x) d\alpha(x) = c_1m(\Delta_1) + \cdots + c_nm(\Delta_n).$$

The definition of the general integral given in the paper referred to above depends on a class  $T_0$  of functions for which the integrals are supposed to be already defined.

We specify this class  $T_0$  to be the class of functions just mentioned, linear combinations of functions of type  $\varphi$ . Evidently a multiple of the modulus of a function, and the sum of two functions of this class is of the same class. Hence  $T_0$  satisfies the required conditions. For such functions

$$(C) \quad \int cf(x) d\alpha(x) = c \int f(x) d\alpha(x),$$

$$(A) \quad \int (f_1 + f_2) d\alpha = \int f_1 d\alpha + \int f_2 d\alpha,$$

$$(M) \quad |\int f d\alpha| \leq \max |f| \times \Sigma_n (\text{variation of } \alpha \text{ on } \Delta_n).$$

It is only necessary to prove that postulate (L) is satisfied. This states that if  $f_1, f_2, \dots$  is a non-increasing sequence of functions of class  $T_0$  which approaches 0 everywhere as a limit, then

$$\lim \int f_n d\alpha = 0.$$

Since  $f_n \leq f_1$ , and  $f_1$  differs from 0 only over a finite set of intervals  $\Delta_1, \Delta_2, \dots, \Delta_k$  each contained within a  $\delta$  of  $CK$ ,

$$\int f_n d\alpha = \int_{\Delta_1} f_n d\alpha + \int_{\Delta_2} f_n d\alpha + \cdots + \int_{\Delta_k} f_n d\alpha.$$

But in each  $\Delta_i$  ( $i = 1, 2, \dots, k$ ),  $\alpha(x)$  is of limited variation so that, by the classical theory of the Stieltjes integral,

$$\lim \int_{\Delta_i} f_n d\alpha = 0.$$

Therefore all the required conditions for the definition of the integral are satisfied, and the definition can be extended, as in the paper to which we have referred, so as to include all integrands,  $f(x)$ , summable with respect to  $\alpha(x)$ .

**3. Standard summable function.** Every interval  $\Delta$  is contained strictly within a  $\delta$  of  $CK$ , and therefore every corresponding  $\varphi$  is 0 at points of  $K$ , and approaches 0 from either side as a point of  $K$  is approached.

A function of class  $T_1$  is the limit of a non-decreasing sequence of functions of class  $T_0$  and a summable function is less than a function of class  $T_1$  and greater than the negative of such a function. Consequently all summable functions must, by definition, vanish at every point of  $K$ . But we can prove more than this. Suppose that the endpoint of an interval  $\delta$  is improper in the direction in which the interval lies. It will be shown that 0 is a sublimit of any summable function  $f(x)$  as  $x$  approaches the endpoint along the interval. Suppose that the point  $x = b$  is the right-hand endpoint of an interval  $\delta$ , and that  $b$  is improper on the left. Let  $a$  be a point within  $\delta$  at which  $\alpha$  is continuous, and let  $\omega(x)$  be the variation of  $\alpha(x)$  between  $a$  and  $x$ . Since  $b$  is improper on the left,  $\omega(x)$  is unbounded as  $x$  approaches  $b$ . If  $f$  is summable with respect to  $\alpha$ , it is also summable with respect to  $\omega$  and its modulus is also summable. Thence

$$\lim \int_a^x f(x) d\omega(x) \quad (x \neq b)$$

must exist, where  $f(x) \geq 0$ . If no sublimit of  $f$  is 0, the lower limit of  $f(x)$  must be positive. Let it be  $l > 0$ . Then an interval  $c, b$  can be found within which  $f(x) > l/2$ . Hence

$$\int_a^x f(x) d\omega(x) \geq l/2 [\omega(x) - \omega(c)],$$

which increases without limit as  $x$  approaches  $b$ .

If  $\alpha(x)$  is sufficiently irregular,  $K$  may consist of every point, and in this case the only summable function will be that which is identically 0, so that for some functions  $\alpha(x)$  the definition of the integral will be valueless. But in many problems although  $\alpha(x)$  is not of limited variation when the whole interval is taken, it is of limited variation when a set is eliminated by a covering set of intervals.

Now although every summable function must vanish at every point of  $K$ , a function can be found which is non-negative and summable, which vanishes nowhere except on points of  $K$ , and which has 0 for a sublimit only as  $x$  approaches an endpoint which is improper on the  $\delta$ -interval side. Such a function plays the part of the function  $h(x) = 1$  in the case of the ordinary Stieltjes integral.\* Within each interval  $\delta_n$  of the set  $CK$  choose a point  $P_n$  ( $x = x_n$ ) at which  $\alpha(x)$  is continuous. Using  $P_n$  as a base we can define a variation function  $\omega_n(x)$  for every point  $x$  in  $\delta_n$ , which is 0 at  $P_n$  and non-decreasing as we proceed from  $P_n$  in either direction.

\* Cf. P. J. Daniell, these Annals, vol. 21 (1920), p. 203.

If  $x$  belongs to  $\delta_n$ , define

$$\begin{aligned}\beta(x) &= \omega_n(x + 0) & x > x_n, \\ &= \omega_n(x - 0) & x < x_n, \\ &= 0 & x = x_n.\end{aligned}$$

Since  $\alpha(x)$  is continuous at  $x = x_n$ ,

$$\omega_n(x + 0) = \omega_n(x - 0) = 0.$$

Define

$$\begin{aligned}h(x) &= 0 \text{ if } x \text{ belongs to } K, \\ &= 1/[2^{2n} + \beta^2(x)] \text{ if } x \text{ belongs to } \delta_n.\end{aligned}$$

We assert that  $h(x)$  is summable with respect to  $\alpha(x)$ . It is evidently non-negative and differs from 0 except on  $K$ . It is also the limit of a sequence of functions of class  $T_0$ . Let  $\Delta_n$  be an interval of length  $d$  of which  $x_n$  is the left-hand endpoint contained within a  $\delta_n$ , and such that  $\alpha$  is continuous at  $x_n + d$  (we have already chosen  $x_n$  to be a point of continuity of  $\alpha$ ). Let

$$\begin{aligned}h_n(x) &= h(x) \text{ on } \Delta_n, \\ &= 0 \text{ otherwise.}\end{aligned}$$

Then  $h_n(x)$  is summable with respect to  $\alpha$  and

$$\int h_n(x) d\alpha(x) = \int_{\Delta_n} h(x) d\alpha(x).$$

The latter is almost an ordinary Stieltjes integral, for on  $\Delta_n$   $\alpha(x)$  is of limited variation and  $h(x)$ , although not continuous, is monotone and bounded.

$$\begin{aligned}\int h_n(x) |d\alpha(x)| &\leq \int_{x_n}^{x_n+d} h(x) d\omega(x) \\ &= \int_{x_n}^{x_n+d} h(x) d\beta(x).\end{aligned}$$

Here  $\int h |d\alpha|$  denotes the modular integral of  $h$  corresponding to  $\int h d\alpha$ , and the second inequality follows from the fact that at every point of continuity of  $\alpha$ ,  $\omega$  and  $\beta$  coincide in value. Let us now make the Lebesgue transformation,\*

$$\beta(x) = t.$$

At a discontinuity of  $\alpha$ , and therefore of  $\beta$ , there is an interval of values of  $t$  corresponding to the one value of  $x$ , but over this discontinuity  $x$ ,

$$\begin{aligned}\int_x h d\beta &= \frac{\beta(x + 0) - \beta(x - 0)}{2^{2n} + \beta^2(x)} \\ &= \frac{\beta(x + 0) - \beta(x - 0)}{2^{2n} + \beta^2(x + 0)} \\ &< \int_{\beta(x-0)}^{\beta(x+0)} \frac{dt}{2^{2n} + t^2}.\end{aligned}$$

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\* H. Lebesgue, Comptes Rendus, vol. 150, p. 86.

Therefore, finally,

$$\begin{aligned} \int h_n(x) |d\alpha(x)| &\leq \int_0^{\beta(x_n+d)} \frac{dt}{2^{2n} + t^2} \\ &< \pi/2^{n+1}. \end{aligned}$$

Similarly for an interval  $\Delta$  of which  $x_n$  is the right-hand endpoint,

$$\int h_n(x) |d\alpha(x)| < \pi/2^{n+1}.$$

If then  $h'(x) = h(x)$  on each of a countable set of intervals  $\Delta$  contained within the  $\delta_n$  of  $CK$ , and 0 otherwise,

$$\begin{aligned} \int h'(x) |d\alpha(x)| &< \pi(\tfrac{1}{2} + \tfrac{1}{4} + \dots) \\ &= \pi. \end{aligned}$$

But  $h$  is the limit of a non-decreasing sequence of functions of type  $h'$  and therefore, by a theorem in the paper to which we referred at the beginning,  $h$  is summable with respect to  $\alpha$ . Many other such functions can easily be constructed by the reader.

Now if  $f(x)$  is any function, summable with respect to  $\alpha$ , it must vanish wherever  $h$  vanishes, so that a function  $\varphi$  can be found such that

$$f(x) = \varphi(x)h(x).$$

If  $\Delta$  is any interval, define its "mass" as

$$m_1(\Delta) = \int_{\Delta} h(x) d\alpha(x).$$

Then  $m_1(\Delta)$  is an additive function of intervals, by means of which we can define the more usual type of integral (Radon-Young integral).

If  $\psi$  is a step-function (that is, constant over each of a finite number of sub-intervals), evidently

$$\int \psi(x) dm_1(e) = \int \psi(x) h(x) d\alpha(x),$$

where  $e$  denotes the variable set of integration. Hence, step by step, it can be proved that if  $\psi$  is summable with respect to  $m_1$ ,  $\psi h$  is summable with respect to  $\alpha$  and vice versa, and that

$$\int \psi(x) dm_1(e) = \int \psi(x) h(x) d\alpha(x).$$

Finally, therefore, if  $f(x)$  is summable with respect to  $\alpha(x)$ , it can be expressed in the form  $\psi(x)h(x)$ , where  $\psi(x)$  is summable with respect to  $m_1(e)$  and

$$\int f(x) d\alpha(x) = \int \psi(x) dm_1(e).$$

This transforms the general type of integral considered to one of the Radon-Young type (extension of the Stieltjes integral by the methods of Lebesgue).

In the particular case given in the paper on a General Form of Integral and mentioned in the introduction to the present paper,

$$\alpha(x) = \log x,$$

the set  $K$  consists of the point  $x = 0$  only, and

$$h(x) = 1/[4 + (\log x)^2].$$

If  $0 \leq x \leq 1$ ,

$$\frac{x}{4} \leq \frac{1}{4 + (\log x)^2}.$$

Hence  $x/4$  is also summable with respect to  $\log x$  and  $x$  satisfies the requirements for an  $h(x)$  (in the interval  $0, 1$ ). Here

$$\begin{aligned} m_1(\Delta) &= \int_{\Delta} x d \log x \\ &= \text{length of } \Delta, \end{aligned}$$

and if  $f(x)/x$  is continuous in the interval,  $f(x)$  is summable with respect to  $\log x$ .

Another example is obtained in the following way: Let  $E$  be a perfect set contained in the interval  $J = (0, 1)$ . Define  $\alpha(x)$  as equal to  $x$  at all points of the intervals making up the set  $J - E$ , and at all irrational points whatever, but equal to 0 otherwise (that is, at rational points not belonging to  $J - E$ ). Then the proper intervals  $\delta_n$  consist of the open intervals forming  $J - E$ , while the set  $K$  consists of the points  $x \leq 0$ , the set  $E$ , and the points  $x \geq 1$ . If  $f_1(x)$  is summable in the usual sense on the interval  $J$ , and if

$$\begin{aligned} f(x) &= 0 \text{ on } K \\ &= f_1(x) \text{ on } J - E = \Sigma \delta_n, \end{aligned}$$

then  $f$  is summable with respect to  $\alpha$  and

$$\int f(x) d\alpha(x) = \Sigma_n \int_{\delta_n} f_1(x) dx.$$

**4. Integration of sets.** According to the paper on a General Form of Integral to which we have already referred, an integral can be defined, or at least extended, by means of certain simple processes such as addition, taking the greater or less of two functions and taking the limit of a monotone sequence. These processes have their analogies in the theory of sets of points, or of more general classes. We recall briefly the main principles of such processes.

There is assumed to be given a fundamental set  $J$  of elements,  $p$ . In this set are contained all the sets considered. The complement of  $J$  is the null set  $\theta$  containing no elements.  $E_1 E_2$ , the *product* of  $E_1$ ,  $E_2$ , is the set of elements belonging to both. It corresponds both to an alge-

braic product and to the "logical product" (the lesser of two numbers).  $E_1 + E_2$ , the *sum* of  $E_1$ ,  $E_2$ , is the set of points belonging to either and corresponds to the logical sum (the greater of two numbers) while when  $E_1$ ,  $E_2$  have no point in common it also corresponds to an algebraic sum. A vital distinction between products and sums in the theory of sets and in algebra is to be noted. The addition (multiplication) of a collection of algebraic numbers is impossible unless the collection has a power not greater than that of a denumerable infinity, and even then an infinite series (product) may not converge. But the sum (product) of any number of sets contained in  $J$  consists of the elements belonging to any one (every one) of the sets and this sum (product) is contained in  $J$ . If every element of  $E_1$  is an element of  $E_2$ , we say that  $E_1 < E_2$  (in particular  $E < E$ ) and then  $E_2 - E_1$  is the set  $E_2 CE_1$ , where  $CE_1$  is the set complementary to  $E_1$ . Subtraction is a useful process in the theory of sets but a dangerous one. For example, it is not, in general, true that  $(A - B) + C = (A + C) - B$ . Again there are no fractional or negative sets.

If  $E_1, E_2, \dots$  is a sequence of sets, we can form the sets,

$$F_n = E_n + E_{n+1} + \dots \quad (n = 1, 2, \dots).$$

$F_1, F_2, \dots$  is a decreasing sequence of sets whose limit  $F$  is the "complete limit" (according to Borel) of the sequence  $[E_n]$ . The limit  $F$  is defined as  $F = F_1 F_2 \dots$ , the set of elements belonging to every  $F_n$ .

If an element belongs to a finite or zero number of the sets,  $E_n$ , it is not contained in some  $F_n$  and is therefore not in  $F$ . If an element belongs to an infinity of the sets  $E_n$ , it belongs to every  $F_n$  and therefore to  $F$ . Hence  $F$  is the set of elements belonging to an infinity of the sets  $E_n$ .  $F$  may be called the "upper limit" of the sequence  $E_n$ . Similarly the lower limit or "restricted limit" (Borel)  $G$  is

$$\begin{aligned} G &= G_1 + G_2 + \dots \\ G_n &= E_n E_{n+1} \dots \end{aligned}$$

$G$  is the set of elements belonging to all but a finite number of the sets  $E_n$ . If  $F = G$ , the sequence  $[E_n]$  is said to converge to the limit  $F$ . This occurs when every element which belongs to an infinity of the  $E_n$  belongs to all but a finite number of them.

**5. Set-functions.** Let  $s$  be a real number. If to each value of  $s$  there corresponds a set  $F(s)$  of elements  $p$ , we say that  $F(s)$  is a set-function of  $s$ . The first analogy with the Stieltjes integral which suggests itself is the integral of  $F(s)$  with respect to  $E(s)$ , where  $F(s)$  is a continuous set-function and  $E(s)$  a set-function of limited variation. But such an analogy

is valueless. On the one hand, even if we define a modular difference of classes, the sum of any number of such modular differences is always contained in  $J$  and a "variation" would be *always* limited.

On the other hand, a *continuous set-function must be constant*. A set-function  $F(s)$  is said to be continuous at  $s = a$  if, whenever  $\lim s_n = a$ ,  $\lim F(s_n)$  exists and equals  $F(a)$ . Let  $F(s)$  be the function assumed to be continuous for all values of  $s$  ( $-\infty < s < +\infty$ ) and let  $S(p)$  be the set of *real numbers*  $s$  for which  $p$  is an element of  $F(s)$ . When  $\lim s_n = s$ ,  $\lim F(s_n) = F(s)$ . Therefore if  $s_1, s_2, \dots$  belong to  $S(p)$ ,  $p$  belongs to  $F(s_n)$  for all  $n$  and therefore to  $F(s)$ . Thus  $S(p)$  is a closed set of real numbers. But  $CS(p)$  corresponds in the same way to  $J - F(s)$ , which is also continuous as a set-function, and  $CS(p)$  must also be closed. But a set of real numbers and its complement cannot both be closed unless one is the set of all numbers, the other of none. In consequence an element  $p$  either belongs to  $F(s)$  for all  $s$  or for no values of  $s$ , and  $F(s)$  is a constant set, the same for all  $s$ .

It is necessary to proceed in a different manner, using the essential distinctions between numbers and sets. Let  $E(s)$  be an increasing set-function (there is no distinction between increasing and non-decreasing since any set is less than—and greater than—itsself in the sense of inclusion). Then if  $s_1 < s_2$ ,  $E(s_1) < E(s_2)$ . If  $s_1, s_2, \dots$  approaches  $s$  from below,  $E(s_n)$  is an increasing sequence of sets possessing a limit. Also this limit is unique and may be called  $E(s - 0)$ . Similarly  $E(s + 0)$  can be defined.

Define

$$\delta E(s) = E(s + 0) - E(s - 0).$$

If  $e(s)$  is any set-function of  $s$ , let

$$\sigma e(s) \quad (S)$$

denote the sum of the sets  $e(s)$  for all values of  $s$  belonging to the collection specified by  $S$ . Then if  $F(s)$  is *any* set-function and  $E(s)$  an increasing set-function, we define

$$\int F(s) dE(s) = \sigma[F(s) \delta E(s)] \quad (-\infty < s < +\infty).$$

This "set-integral" possesses certain interesting properties. For example, if  $F(s), G(s)$  are two set-functions,

$$\int (F + G) dE = \int F dE + \int G dE.$$

For, omitting the variable  $s$ ,

$$\begin{aligned} \sigma[(F + G) \delta E] &= \sigma[F \delta E + G \delta E] \\ &= \sigma[F \delta E] + \sigma[G \delta E]. \end{aligned}$$

Also

$$\int F G dE = (\int F dE)(\int G dE).$$

For if  $s < t$ ,

$$\begin{aligned}\delta E(s) &= E(s+0) - E(s-0) \\ &< E(s+0) \\ &< E(t-0).\end{aligned}$$

Therefore  $\delta E(s)$ ,  $\delta E(t)$  have no element in common, and, by symmetry,

$$\begin{aligned}\delta E(s) \cdot \delta E(t) &= \delta E(s) \quad (s = t), \\ &= \theta \quad (s \neq t). \\ (\int F dE)(\int G dE) &= \sigma_s[F(s)\delta E(s)] \cdot \sigma_t[G(t)\delta E(t)] \\ &= \sigma_{s,t}[F(s)G(t)\delta E(s)\delta E(t)] \\ &= \sigma_s[F(s)G(s)\delta E(s)] \\ &= \int F G dE.\end{aligned}$$

If it is recalled that the product of a set into itself is equal to itself, the above inequality can be expressed in a form which reminds one of the Schwarz inequality in ordinary integration, this form being

$$(\int F G dE)^2 = (\int F^2 dE)(\int G^2 dE).$$

But in this theory of sets, infinite addition and multiplication can be as readily handled as finite processes, and by the same reasoning as before

$$\begin{aligned}\int (F_1 + F_2 + \dots) dE &= \int F_1 dE + \int F_2 dE + \dots, \\ \int F_1 F_2 \dots dE &= (\int F_1 dE)(\int F_2 dE) \dots.\end{aligned}$$

It follows from these equalities and the definition of a limit that

$$\int \lim F_n(s) dE(s) = \lim \int F_n(s) dE(s).$$

Let  $r^*t$  denote an interval  $r < s \leq t$ , equal to the interval  $(r, t)$  (closed) with the point  $r$  omitted. Let

$$\begin{aligned}F_{r^*t}(s) &= F(s) \quad (r < s \leq t) \\ &= \theta \text{ otherwise.}\end{aligned}$$

We may define

$$\int_{r^*t} F(s) dE(s) = \int F_{r^*t}(s) dE(s).$$

Then

$$\int_{r^*t} F dE + \int_{t^*u} F dE = \int_{r^*u} F dE$$

Also

$$\begin{aligned}\int_{r^*t} J dE(s) &= \sigma(\delta E(s)) \quad (r < s \leq t) \\ &= E(t+0) - E(r+0).\end{aligned}$$

For consider an element  $p$ . The numbers  $s$  can be placed in one of two classes  $S(p)$ ,  $CS(p)$  according to whether  $p$  belongs to  $E(s)$  or not. Since  $E(s)$  is increasing, any number in  $CS(p)$  is less than any in  $S(p)$  and, if

both classes exist, a "section" is obtained which defines a real number  $s$  (dependent on  $p$ ). In this case  $p$  belongs to  $E(s+0)$  but not to  $E(s-0)$ . Any element  $p$  must therefore satisfy one of three conditions, (a)  $p$  belongs to all  $E(s)$ , or (b)  $p$  belongs to no  $E(s)$ , or (c)  $p$  belongs to  $\delta E(s)$  for some value of  $s$ . If  $p$  belongs to  $E(t+0) - E(r+0)$ , it satisfies neither (a) nor (b), and  $p$  belongs to  $\delta E(s)$  for some  $s$ . *This  $s$  must lie in the interval,  $r < s \leq t$ .* For if  $s \leq r$ ,  $p$  belongs to  $E(s+0)$  which is excluded from  $E(t+0) - E(r+0)$  in  $E(r+0)$ . Similarly if  $s > t$ ,  $p$  is excluded from  $E(s-0)$  which, however, includes  $E(t+0)$ . Then

$$E(t+0) - E(r+0) < \sigma[\delta E(s)] \quad (r < s \leq t).$$

But if  $p$  belongs to some  $\delta E(s)$  ( $r < s \leq t$ ), it belongs to  $E(s+0)$  and therefore to  $E(t+0)$ , while it does not belong to  $E(s-0)$  and therefore not to  $E(r+0)$ . So that

$$\sigma[\delta E(s)] \quad (r < s \leq t) < E(t+0) - E(r+0).$$

This proves the required equality. Evidently intervals  $(r, t)$ ,  $(r, t^*)$ ,  $(r^*, t^*)$  can be handled in the same manner.

**6. Directed continuity.** It has been proved that, if a set-function is continuous, it is constant and its properties are of little interest. But we can obtain a valuable class of functions if we restrict the continuity to be on one side only.

If for all values of  $s$ , the unique limit  $F(s+0)$  exists and is equal to  $F(s)$ , then  $F(s)$  is said to be continuous on the right. Similarly if  $F(s-0) = F(s)$ ,  $F(s)$  is continuous on the left. A set-function is called a step-function if it is constant over each of a finite number of intervals of  $s$ .

**THEOREM.** *A function which is continuous on the right is the limit of a sequence of step-functions.*

For if  $F(s)$  is the given function and if

$$F_n(s) = F[t_n(s)],$$

where  $2^n t_n(s)$  is the least integer not less than  $2^n s$  (the integer equal to or just greater than  $2^n s$ ), then

$$F(s) = \lim F_n(s).$$

Since  $t_n(s)$  is a non-increasing sequence approaching  $s$  from above, if  $s$  is not a terminating fraction in the scale of 2,

$$\lim F_n(s) = F(s+0),$$

and otherwise, after some finite  $n$ ,  $t_n(s) = s$  so that

$$F_n(s) = F(s).$$

The theorem is thus proved. Now if  $t_{i,n} = i2^{-n}$  where  $i$  is a positive or negative integer or zero,

$$\begin{aligned}\int F_n(s) dE(s) &= \sum_i F(t_{i,n}) [\sigma \delta E(s) (t_{i-1,n} < s \leq t_{i,n})] \\ &= \sum_i F(t_{i,n}) [E(t_{i,n} + 0) - E(t_{i-1,n} + 0)].\end{aligned}$$

Since

$$\int F dE = \lim \int F_n dE$$

the following important theorem is an immediate consequence:

**THEOREM.** *If for each value of  $s$ ,  $F(s)$ ,  $E(s)$  are sets of points in one or more dimensions which are  $B$ -measurable (measurable in the sense of Borel); if  $E(s)$  is an increasing set-function and  $F(s)$  continuous on the right, then  $\int F dE$  is also  $B$ -measurable.*

According to our primary definition for any  $F$ , an integral is obtained by an infinite process having the power of the continuum, but we see that, if  $F$  is continuous on the right, the integral can be obtained by passages from finite processes to the limit, processes which do not take the sets beyond the class of  $B$ -measurable sets.

The same result would hold if  $F(s)$  were continuous on the left or if it were continuous on the left or right in each of a countable number of intervals, whose complementary set is a countable number of points ("countable" means zero, finite or denumerably infinite). The theorem also holds if "measurable in the sense of Lebesgue" is substituted for " $B$ -measurable." This theorem has an immediate application to the theory of measurable functions. Let  $E(s)$ ,  $F(s)$  be the sets of points for which  $e(p) < s$ ,  $f(p) < s$ , respectively, where  $e(p)$ ,  $f(p)$  are never-infinite functions of points  $p$  in one or more dimensions. If  $e$ ,  $f$  are measurable (in either sense, this sense being retained throughout), so are  $E(s)$ ,  $F(s)$  measurable for each  $s$ . These sets are increasing and continuous on the left. For if  $f(p) < s$ , after some finite value of  $n$ ,  $f(p) < s - 2^{-n}$  and  $p$  belongs to  $F(s - 2^{-n})$ . If  $f(p) \geq s$ ,  $p$  belongs to no set  $F(s - 2^{-n})$ . The set  $G(s)$  of points for which  $e(p) + f(p) < s$  consists of the sum for all  $t$  of the sets where simultaneously  $e(p) = t$ ,  $f(p) < s - t$ .

Now the set where  $e(p) = t$  is the set  $\delta E(t)$  and therefore

$$\begin{aligned}G(s) &= \sigma_t F(s - t) \delta E(t) \\ &= \int F(s - t) dE(t).\end{aligned}$$

Considered as a function of  $t$ ,  $F(s - t)$  is continuous on the right and, by our theorem,  $G(s)$  is measurable and

$$g(p) = e(p) + f(p)$$

is measurable in the same sense as  $e(p) \cdot f(p)$ . This proves that the sum of two never-infinite measurable functions is measurable. A similar

proof is possible for the product of two measurable functions, although this case can be considered more readily by a combination of the previous theorem with one proving that the square of a measurable function is measurable. It may be thought for a moment that if  $E(s)$ ,  $F(s)$  are measurable for each  $s$  and if  $E(s)$  is an increasing set-function, then

$$\int FdE$$

is measurable without further restrictions on  $F$ , but this is not true. For example, let  $E(s)$  be the set of real numbers less than  $s$ , so that  $\delta E(s)$  is the number (point)  $s$  itself. Let  $f(t)$  be some non-measurable function of  $t$ . Denote by  $F(s)$  the set of real numbers less than  $f(s) + s - a$ . Then  $E(s)$ ,  $F(s)$  are certainly measurable for every  $s$ .  $F(s)\delta E(s)$  will be that number, if it exists, which is simultaneously equal to  $s$  and less than  $f(s) + s - a$ . Hence

$$\int F(s)dE(s)$$

is the set of numbers such that  $s < f(s) + s - a$ , that is to say, the set of numbers  $s$  for which  $f(s) > a$ . But  $f(t)$  is non-measurable and therefore for some value of  $a$  the above set is non-measurable.

It would be interesting to study more closely the conditions which must be laid on  $F(s)$  in order that the integral should be measurable. It is probably unnecessary that  $F(s)$  should be continuous in one direction even in a number of intervals.

**7. Geometrical illustration.** It is helpful in a study of this integral to have in mind an illustration which is as follows: Let  $E(s)$  be a set of values of the real variable  $x$  for each  $s$ . In the plane use Cartesian co-ordinates  $Ox$  horizontal,  $Os$  vertical where  $Oy$  is usually drawn. Through each point on  $Os$  draw a horizontal line and mark on it the points whose  $x$ -coördinates belong to  $E(s)$ . Then corresponding to the set-function  $E(s)$  there is a plane set  $e$ . Similarly to  $F(s)$  corresponds a plane set  $f$ .

If  $E(s)$  is an increasing set-function,  $e$  consists of the points belonging to a collection of vertical lines which are unbounded above. If  $d$  is the plane set corresponding to  $\delta E(s)$ ,  $d$  will consist of the lower bounds (where they exist) of these vertical lines. The integral  $\int FdE$  consists of the projection on the  $x$ -axis of the plane set  $fd$  common to  $f$  and  $d$ . If  $e(x)$  is some function of  $x$  and if  $E(s)$  is the set of values of  $x$  for which  $e(x) < s$ , then the corresponding plane set  $e$  is the set of points above but not including the "graph" of  $s = e(x)$ . The plane set  $d$  corresponding to  $\delta E(s)$  is the set of points of which the "graph" consists.

If now  $F(s)$  is the set of values of  $x$  for which  $f(x) < s$ , the plane set  $f'$  corresponding to  $F(s - t)$  (considered as a set-function of  $t$ ) is obtained by taking the image of the curve  $t = f(x)$  (in the  $xOt$  plane) with respect

to the  $x$ -axis, moving it up a constant distance  $s$ , and then by choosing all the points in the plane below but not including this transformed curve. The set  $G(s)$  corresponding to  $g(x) = e(x) + f(x)$  is the set of values of  $x$  for which the curve  $t = e(x)$  falls strictly below the curve  $t = s - f(x)$ , that is to say, for which  $e(x) + f(x) < s$ .

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